MAT 1348 3X – Test # 3 – Spring/Summer 2016

Full Name:____________________________________

Student Number:______________________________

Cellular phones, unauthorized electronic devices or course notes (unless an open-book exam) are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

By signing below, you acknowledge that you have ensured that you are complying with the above statement.

Signature:______________________________

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Question	Possible Points	Points Obtained
# 1	5	
# 2	5	
# 3	5	
# 4	5	
# 5	5	
Total	25	

Instructions:

- Print your name and student number on the first two pages.
- Verify that your copy of the test has all of its 9 pages.
- You must answer all questions. There are 5 questions for a total of 25 points.
- Write the solutions to the questions in the space provided. You may use the back of the pages if necessary.
- This a closed book test, no course notes are permitted.
- Calculators are permitted.

SHOW ALL YOUR WORK

1. (**5 pts**)

   (a) Let A, B be two sets, and the function $f : A \to B$. Give the definition of each of the following terms:

      i. f is injective.

      ii. f is surjective.

   (b) Is the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by:

   $$f(m, n) = (n + 6, 2 - m)$$

      i. injective?

      ii. surjective?

   Justify your answer

Solution:

(a) i. A function f is said to be injective if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

   ii. A function f is said to be surjective if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

(b) i. Yes,

$$f(m, n) = f(x, y) \Rightarrow (n + 6, 2 - m) = (y + 6, 2 - x) \Rightarrow \begin{cases} n + 6 = y + 6 \\ 2 - m = 2 - x \end{cases}$$

$$\Rightarrow \begin{cases} n = y \\ m = x \end{cases} \Rightarrow (m, n) = (x, y)$$

   ii. Yes, Given (a, b) in the codomain,

$$f(m, n) = (a, b) \Rightarrow (n + 6, 2 - m) = (a, b) \Rightarrow \begin{cases} n + 6 = a \\ 2 - m = b \end{cases} \Rightarrow \begin{cases} n = a - 6 \\ m = 2 - b \end{cases}$$

   Since for any (a, b) in the codomain, $(2 - b, a - 6)$ is in the domain and $f(2 - b, a - 6) = (a, b)$, f is surjective.

2. (**5 pts**)

   (a) Show that the function $f(x) = e^x$ from the set of real numbers to the set of real numbers is not invertible.

   (b) Which of the following will result in modifying the function to be invertible:

      i. if the domain is restricted to the set of positive real numbers.

      ii. if the codomain is restricted to the set of positive real numbers.

   (c) Given the changes made in the previous part, give the inverse of $f(x)$.

Solution:

   (a) To be invertible, the function must be a bijection. However, since -5 is in the codomain but $e^x > 0$ for any x in the domain, there is no pre-image for -5. Thus, f is not surjective. Hence, the function is not bijective.

   (b) (ii)

   (c)

$$\begin{aligned} y &= f(x) \\ y &= e^x \\ \ln(y) &= x \\ \ln(x) &= y \\ f^{-1}(x) &= y \end{aligned}$$

   Thus, $f^{-1}(x) = \ln(x)$ is the inverse.

3. (**5 pts**) Define a relation on the set of integers $\mathbb{Z}$ by

$$x \; R \; y \Leftrightarrow x - y \text{ is a multiple of 3.}$$

(a) Show that R is an equivalence relation on $\mathbb{Z}$.

(b) Let $A = \{-5, -2, -1, 0, 2, 3, 5, 8, 9, 11\} \subseteq \mathbb{Z}$. Give a partition of A defined by the equivalence relation R given in the previous part.

Solution:

(a)
- Reflexive: For any $x \in \mathbb{Z}$, since $x - x = 0 = 0 \cdot 3$, we know that $x - x$ is a multiple of 3. Thus (x, x) in R.
- Symmetric: Suppose (x, y) in R. Hence $x - y$ is a multiple of 4. Thus, there exists an integer k such that $x - y = k \cdot 3$. Since $y - x = (-k) \cdot 3$ is a multiple of 3, we have shown that (y, x) is in R.
- Transitive: Suppose (x, y) and (y, z) in R. Thus, there exists integers k and l such that $x - y = k \cdot 3$ and $y - z = l \cdot 3$. Since $x - z = (x - y) + (y - z) = k \cdot 3 + l \cdot 3 = (k + l) \cdot 3$, we have shown that (x, z) is in R.

(b) Consider

$$A_1 = [-5]_R = \{-5, -2\},$$
$$A_2 = [-1]_R = \{-1, 2, 5, 8, 11\},$$
$$A_3 = [0]_R = \{0, 3, 9\}.$$

Thus A_1, A_2, and A_3 form a partition of A.

4. (**5 pts**) A and B are two finite sets with $|A| = 4$ and $|B| = 6$.

   (a) How many relations are there from A to B?

   (b) How many functions $f : A \to B$ are there?

   (c) How many **injective** functions $f : A \to B$ are there?

   Solution:

   (a) $|P(A \times B)| = 2^{|A \times B|} = 2^{|A| \cdot |B|} = 2^{(4)(6)} = 2^{24} = 16,777,216$

   (b) To be a function first we pick an input and assign to it a possible output. There are 4 possible inputs and there are 6 possible outputs. Thus, by the product rule there are $6^4 = 1,296$ possible functions.

   (c) To be an injective function we must choose for each input a distinct output. So for the first input there are 6 possible choices, for the second there are 5 choices, for the third there are 4 choices and there are 3 choices for the last input. Thus by the product rule there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ possible injective functions.

5. (**5 pts**) A computer password is formed by 4 or 5 characters. Each character must be an alphanumerical symbol (a lowercase English letter or one of the 10 digits). How many passwords would contain at least one letter.

Solution:

- Let P be the total number of passwords formed by 4 or 5 characters which contain at least one letter.
- Let P_4 be all the passwords formed by 4 characters which contain at least one letter.
- Let P_5 be all the passwords formed by 5 characters which contain at least one letter.
- Then, by the sum rule we have $P = P_4 + P_5$.
- To find P_4 we compute the total number of possible 4 character passwords, which is 36^4, and subtract the number of 4 character passwords which contains no letters, which is 10^4.
- Thus, $P_4 = 36^4 - 10^4 = 1,679,616 - 10,000 = 1,669,616$.
- Similarly, $P_5 = 36^5 - 10^5 = 60,466,176 - 100,000 = 60,366,176$.
- Hence,
$$P = P_4 + P_5 = 62,035,792.$$

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